24.08.2016

<u>Lecture 14 and 15</u> Metric and Hilbert Spaces ① Univ. Melb.

<u>Proposition</u> Let $(X, \mathcal{T})$ be a topological space. If X is cover compact then X is sequentially compact.

<u>Proof</u> Assume X is not sequentially compact.

Then there exists a sequence $(a_1, a_2, \ldots)$ with no cluster point.

Thus, if $z \in X$ then there exists $N \in \mathcal{N}(z)$ such that $\mathrm{Card}\{j \mid a_j \in N\}$ is finite.

To show: X is not cover compact.

To show: There exists an open cover $\mathcal{S}$ with no finite subcover.

For $x \in X$ let $V_x \in \mathcal{T}$ with

$x \in V_x$ and $\mathrm{Card}\{j \in \mathbb{Z}_{>0} \mid a_j \in V_x\}$ finite.

Let $\mathcal{S} = \{V_x \mid x \in X\}$

To show: $\mathcal{S}$ has no finite subcover.

Assume $\ell \in \mathbb{Z}_{>0}$ and $S_1, S_2, \ldots, S_\ell \in \mathcal{S}$ such that $S_1 \cup S_2 \cup \cdots \cup S_\ell \supseteq X$

Let $k_j \in \mathbb{Z}_{>0}$ be such that

if $n \in \mathbb{Z}_{>k_j}$ then $a_n \notin S_j$.

Let $k = \max\{k_1, k_2, \ldots, k_\ell\}$.

Then, if $n \in \mathbb{Z}_{>k}$ then $a_n \notin S_1 \cup \cdots \cup S_\ell$.

This is a contradiction to $S_1 \cup \cdots \cup S_\ell \supseteq X$.

So $\mathcal{S}$ has no finite subcover.

So X is not cover compact. //

<u>Proposition</u> Let (X,d) be a metric space.
Assume (X,d) is ball compact.
If $A \subseteq X$ then (A,d) is ball compact.

<u>Proof</u> Assume X is ball compact and $A \subseteq X$.

To show: A is ball compact.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exist $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in A$ such that

$$B_\varepsilon(a_1) \cup B_\varepsilon(a_2) \cup \cdots \cup B_\varepsilon(a_\ell) \supseteq A.$$

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exist $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in A$ such that

$$B_\varepsilon(a_1) \cup B_\varepsilon(a_2) \cup \cdots \cup B_\varepsilon(a_\ell) \supseteq A.$$

Let $m \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_m \in X$ such that

$$B_{\frac{\varepsilon}{2}}(x_1) \cup \cdots \cup B_{\frac{\varepsilon}{2}}(x_m) \supseteq X.$$

Let ℓ be the number of $B_{\frac{\varepsilon}{2}}(x_j)$ with $B_{\frac{\varepsilon}{2}}(x_j) \cap A \neq \emptyset$.
For each $B_{\frac{\varepsilon}{2}}(x_j)$ with $B_{\frac{\varepsilon}{2}}(x_j) \cap A \neq \emptyset$ let $a_j \in B_{\frac{\varepsilon}{2}}(x_j) \cap A$.

Then
$$\begin{aligned} B_\varepsilon(a_1) \cup \cdots \cup B_\varepsilon(a_\ell) &\supseteq \left(B_{\frac{\varepsilon}{2}}(x_1) \cap A\right) \cup \cdots \cup \left(B_{\frac{\varepsilon}{2}}(x_m) \cap A\right) \\ &\supseteq X \cap A = A, \end{aligned}$$

since $B_\varepsilon(a_j) \supseteq B_{\frac{\varepsilon}{2}}(x_j)$

. So A is ball compact. //

Proposition Let (X,d) be a metric space.

If X is cover compact then X is ball compact.

Proof Assume X is cover compact.

To show: X is ball compact

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exist $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots x_\ell \in X$ with $B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_\ell) \supseteq X$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Then $\mathcal{S} = \{ B_\varepsilon(x) \mid x \in X \}$ is an open cover of X.

Since X is cover compact then $\mathcal{S}$ has a finite subcover.

So there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots x_\ell \in X$ such that $B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_\ell) \supseteq X$.

So X is ball compact //.

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<u>Proposition</u> Let (X,d) be a metric space.

If X is ball compact then X is bounded.

<u>Proof</u> Assume X is ball compact.

To show: X is bounded.

To show: There exists $a\in X$ and $M\in\mathbb{R}_{>0}$ such that $X\subseteq B_M(a)$.

Using that X is ball compact there exists $\ell\in\mathbb{Z}_{>0}$ and $x_1, x_2, \ldots x_\ell\in X$ with

$$B_1(x_1)\cup B_1(x_2)\cup\cdots\cup B_1(x_\ell)\supseteq X.$$

Let $a=x_1$ and $M=\max\{d(a_1,a_1),\ldots,d(a_1,a_\ell)\}+2$.

To show: $X\subseteq B_M(a)$.

To show: If $x\in X$ then $x\in B_M(a)$.

Assume $x\in X$.

To show: $x\in B_M(a)$

To show: $d(x,a)<M$

Let x_j be such that $x\in B_1(x_j)$. Then

$$\begin{aligned} d(x,a)=d(x,a_1)&\le d(x,x_j)+d(a_j,a_1)\\ &< 1+(M-2)=M-1<M. \end{aligned}$$

So $x\in B_M(a)$

So X is bounded. //